MATH2060B Tutorial 10

Topic: Constructing hyperbolic cosine and sine

Q6 To show existence of functions $c, s : \mathbb{R} \to \mathbb{R}$ such that

(i) $c''(x) = c(x)$, $s''(x) = s(x)$.

(ii) $c(0) = 1$, $c'(0) = 0$, $s(0) = 0$, $s'(0) = 1$

Moreover, c, s satisfy $c'(x) = s(x)$, $s'(x) = c(x)$ $\forall x \in \mathbb{R}$.

Sol

① Setup: Define sequences (c_n) and (s_n) inductively

$$c_1(x) = 1 \qquad s_1(x) = x$$

$$s_n(x) = \int_0^x c_n(t)\,dt, \qquad c_{n+1}(x) = 1 + \int_0^x s_n(t)\,dt$$

By induction, all these functions are continuous on $\mathbb{R}$, and integrable over any bounded interval (i.e. well-defined). By Fundamental Thm of Calculus, all are differentiable everywhere and

$$s_n'(x) = c_n(x), \quad c_{n+1}'(x) = s_n(x) \quad \forall x \in \mathbb{R}.$$

In fact, (by induction)

$$c_{n+1}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!}.$$

$$s_{n+1}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$$

② Uniform convergence (over bounded intervals)

Let $A > 0$, and take $m > n > 2A$. Then for $|x| \leq A$

$$|c_m(x) - c_n(x)| = \left| \frac{x^{2n}}{(2n)!} + \frac{x^{2n+2}}{(2n+2)!} + \cdots + \frac{x^{2m-2}}{(2m-2)!} \right|$$

$$\leq \frac{A^{2n}}{(2n)!}\left[1 + \frac{(2n)!}{(2n+2)!}A^2 + \cdots + \frac{(2n)!}{(2m-2)!}A^{2(m-1-n)}\right]$$

$$\leq \frac{A^{2n}}{(2n)!}\left[1 + \left(\frac{A}{2n}\right)^2 + \cdots + \left(\frac{A}{2n}\right)^{2(m-1-n)}\right]$$

$$< \frac{A^{2n}}{(2n)!} \cdot \frac{16}{15} \qquad \left(\because \frac{A}{2n} < \frac{1}{4}\right)$$

Since $\lim_{n\to\infty} \frac{A^{2n}}{(2n)!} = 0$, Cauchy Criterion $\Rightarrow$ c_n converges uniformly on $[-A, A]$, $\forall A > 0$.

In particular, $c_n(x)$ converges pointwise to a function $c: \mathbb{R} \to \mathbb{R}$

$\to$ Uniform convergence $c_n \rightrightarrows c$ on $[-A, A]$, $\forall A > 0$

$\Rightarrow c$ is continuous on $\mathbb{R}$.

$\to$ Note that $c_n(0) = 1 \quad \forall n \Rightarrow c(0) = 1$.

Similarly, when $|x| \leq A$, $m > n > 2A$, since

$$S_m(x) - S_n(x) = \int_0^x c_m(t) - c_n(t)\,dt,$$

we have

$$\left|S_m(x) - S_n(x)\right| \leq \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!} \cdot A \qquad \text{(Cor 7.3.15)}$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

As above, (S_n) converges uniformly on $[-A, A]$, $\forall A > 0$. Its pointwise limit $s: \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $s(0) = 0$.

3 Differentiability Conclusions

For any $A > 0$,

(a) $\quad c_n(x) \rightarrow c(x)$ on $[-A, A]$

(Thm 8.2.3) (b) $\quad c_n'(x) = S_{n-1}(x) \rightrightarrows s(x)$

implies the limit function c is differentiable everywhere with $c'(x) = s(x) \ \forall x \in \mathbb{R}$.

Similarly, $s(x)$ is differentiable everywhere and $s' = c$.

Note that

(i) $\quad c''(x) = s'(x) = c(x), \qquad s''(x) = c'(x) = s(x)$

(ii) $\quad c'(0) = s(0) = 0, \qquad s'(0) = c(0) = 1$

So $c, s: \mathbb{R} \to \mathbb{R}$ satisfy the desired properties. $\square$

Q7 Show that the functions c, s (a) have derivatives of all orders; (b) satisfy the property $(c(x))^2 - (s(x))^2 = 1 \ \forall x \in \mathbb{R}$; and (c) are unique functions satisfying properties (i) and (ii).

Sol (a) follows from $c' = s$ and $s' = c$ (by induction).

(b) Observe that

$$\frac{d}{dx}\left[(c(x))^2 - (s(x))^2\right] = 2c(x)s(x) - 2s(x)c(x) = 0.$$

Therefore $(c(x))^2 - (s(x))^2$ is a constant function whose value is $(c(0))^2 - (s(0))^2 = 1$.

(c) follows from the claim below

Claim Suppose $g: \mathbb{R} \to \mathbb{R}$ is a function satisfying $g'' = g$ and $g(0) = g'(0) = 0$. Then $g \equiv 0$.

Pf By induction, $g^{(n)}$ exists everywhere and $g^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$.

Fix $x \in \mathbb{R}$. Applying Taylor theorem shows that $\forall n \in \mathbb{N}$,

$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \cdots + \frac{g^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{g^{(n)}(c_n)}{n!}x^n, \quad \text{for } c_n \in [0,x]$$

$$= g^{(n)}(c_n) \cdot \frac{x^n}{n!}.$$

Since g and g' are continuous on $[0,x]$, there exists K such that $|g(t)|, |g'(t)| \leq K \ \forall t \in [0,x]$.

Because $g^{(n)} = g$ or $g^{(n)} = g'$, $|g^{(n)}(c_n)| \leq K \ \forall n$.

Therefore $|g(x)| \leq K \cdot \frac{|x|^n}{n!} \ \forall n$, and $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ imply that $g(x) = 0$. $x \in \mathbb{R}$ is arbitrary, so $g \equiv 0$. $\square$

If c_1, c_2 both satisfy (i) and (ii), then

$$(c_1 - c_2)'' = c_1 - c_2, \quad (c_1 - c_2)(0) = 0, \quad (c_1 - c_2)'(0) = 0.$$

Claim $\Rightarrow$ $c_1 - c_2 = 0$ or $c_1 = c_2$. Similarly, s is also unique. //

Remark By existence and uniqueness, we can call c and s the hyperbolic cosine and hyperbolic sine respectively.

Q8 If $f:\mathbb{R}\to\mathbb{R}$ is such that $f''=f$, show that there exist $\alpha,\beta\in\mathbb{R}$ such that $f(x)=\alpha c(x)+\beta s(x)$ for all $x\in\mathbb{R}$.

Sol Consider $h(x)= f(0)c(x)+f'(0)s(x)$.

$$\text{Property (i)} \Rightarrow h''(x)=f(0)c''(x)+f'(0)s''(x)=h(x)$$

$$\text{Property (ii)} \Rightarrow h(0)=f(0), \quad h'(0)=f'(0).$$

So $h-f$ satisfies $(h-f)''=h-f$ and $(h-f)(0)=0=(h-f)'(0)$.

Previous claim $\Rightarrow$ $h-f=0$ or $f(x)=\alpha c(x)+\beta s(x)$ $\forall x$. //

$(\alpha=f(0), \ \beta=f'(0))$

Consider $f_1(x)=e^x$ and $f_2(x)=e^{-x}$. Since $f_1''=f_1$, and $f_2''=f_2$. The above shows that

$$e^x = c(x)+s(x)$$

$$e^{-x}=c(x)-s(x)$$

$$\Rightarrow \quad c(x)=\frac{e^x+e^{-x}}{2}, \quad s(x)=\frac{e^x-e^{-x}}{2} \quad \forall x\in\mathbb{R}.$$